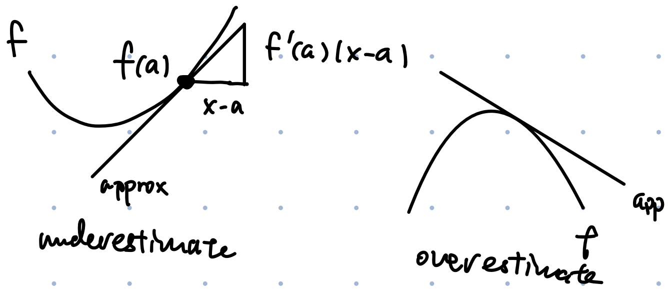
(*) Local approximation and second derivatives (§8.1, 8.2)

- Local approximation: given f differentiable near $x=a$, one has the <u>tangent line approximation</u>:

$$y = f'(a)(x-a) + f(a).$$



- This approximation is only accurate near a: farther x is away from a, the more off is the approximation.
- <u>Q</u>: How do we know our approximation is an over estimate or under estimate?
- Concavity revisited:
- f is <u>concave up</u> near $x=a$, $\begin{cases} \text{if } f'(x) \text{ keeps increasing near } x=a \\ \text{if } f''(x) \text{ is positive near } x=a. \end{cases}$

  → In this case, the tangent line approximation is <u>underestimate</u>.
- f is <u>concave down</u> near $x=a$, $\begin{cases} \text{if } f'(x) \text{ keeps decreasing near } x=a \\ \text{if } f''(x) \text{ is negative near } x=a. \end{cases}$

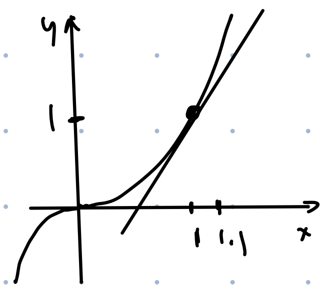
  → In this case, the tangent line approximation is <u>overestimate</u>.
- Meaning of second derivative: how the rate of change roughly changes, when x increase by 1 unit.

<u>Ex</u>. Let $f(x)=x^3$ ① Find the local approximation of f at $x=1$.

② Overestimate or under estimate?

③ Use this approximation to find $f(1.1)$ approximatingly. What's the error?

<u>Sol.n</u>:



① $f'(x) = 3x^2$. $f'(1) = 3$. $f(1) = 1$

- Approximation: $y = f'(1)(x-1) + f(1) = 3(x-1)+1 = 3x-2$.

② $f''(x) = 6x$, $f''(1) = 6 > 0$. So f is concave up near $x=1$.

And the approximation is an underestimate.

③ App: $y = 3(1.1) - 2 = 1.3$. ← Approximation.

Exact: $f(1.1) = 1.1^3 = 1.331$.

Error $= 1.331 - 1.3 = 0.031$ (underestimate).

(*) Derivative rules. (S8.3).

- Very helpful rules: let $f(x)$, $g(x)$ be differentiable.

[1] Sum rule:

$$\frac{d}{dx}(f(x)+g(x)) = \left(\frac{d}{dx}f\right)(x) + \left(\frac{d}{dx}g\right)(x). \quad \Big| \quad (f+g)' = f'+g'$$

[2] Multiple rule: let k be a real number

$$\frac{d}{dx}(k.f(x)) = k\left(\frac{d}{dx}f\right)(x). \quad \Big| \quad (k.f)' = k.f'.$$

[3] Product rule:

$$\frac{d}{dx}(f(x).g(x)) = \left(\frac{d}{dx}f\right)(x).g(x) + f(x).\left(\frac{d}{dx}g\right)(x) \quad \Big| \quad (f.g)' = f'g + fg'.$$

[4] Quotient rule: Assume $g(x) \neq 0$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}. \quad \text{(first diff numerator)}$$

- To practically use those rules, we need to know the derivative of some basic functions.

Thm Let n be a real number. Then

$$\frac{d}{dx}(x^n) = n x^{n-1}. \qquad (+)$$

- When $n=0$, $\frac{d}{dx}(1) = 0$.

Pf (When n is integer.) Induction:

① When $n=1$: $\frac{d}{dx}(x^1) = \lim_{h\to 0} \frac{x+h-x}{h} = 1.$ ✓

② Assume $(+)$ holds for $n=k$ positive integer, then for $n=k+1$,

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x.x^k) = \left(\frac{d}{dx}x'\right).x^k + x'.\frac{d}{dx}(x^k) \qquad \text{(product rule.)}$$

$$= x^k + x.kx^{k-1} = (k+1)x^k. \quad ✓.$$

- So $(+)$ holds for all positive integers.

③ $n=0$: $\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = \lim_{h\to 0}\frac{1-1}{h} = 0.$ ✓.

④ When $n=k$ where k is negative integer. Here $-k$ is positive integer.

So $$\frac{d}{dx}(x^k) = \frac{d}{dx}\left(\frac{1}{x^{-k}}\right) = \frac{(x^{-k}).(1)' - (x^{-k})'.1}{(x^{-k})^2} \qquad \text{(quotient rule)}$$

$$= \frac{x^{-k}.0 - (-k)x^{-k-1}}{x^{-2k}} = \frac{kx^{-k-1}}{x^{-2k}} = kx^{k-1} \quad ✓. \qquad \boxed{\text{Q}}.$$

Ex. Find $\frac{d}{dx}\left(\frac{1+2x^2+3x^3+4x^4}{x^2}\right)$.

Sol.ⁿ $\left(\frac{1+2x^2+3x^3+4x^4}{x^2}\right)' = \left(x^{-2}+2+3x+4x^2\right)'$

(sum)

$$= (x^{-2})' + (2)' + (3x)' + (4x^2)'$$

$$= -2x^{-3} + 0 + 3 + 8x.$$

$$= -2x^{-3} + 3 + 8x.$$

Ex. Find $\frac{d}{dx}\left(\frac{x}{x^3+1}\right)$.

Sol.ⁿ: • $f = x$, $g = x^3+1$.

• $f' = x' = 1$ • $g' = (x^3+1)' = 3x^2$.

• Quotient rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} = \frac{1.(x^3+1) - x(3x^2)}{(x^3+1)^2} = \frac{-2x^3+1}{(x^3+1)^2}$.

Ex. Find $\frac{d}{dx}\left((x^2+1)(x^2-x+1)\right)\Big|_{x=1}$

Sol.ⁿ $\left((x^2+1)(x^2-x+1)\right)' \overset{\text{product}}{=\!=} (x^2+1)'(x^2-x+1) + (x^2+1)(x^2-x+1)'$

$$= 2x(x^2-x+1) + (x^2+1)(2x-1)$$

$$\frac{d}{dx}\left((x^2+1)(x^2-x+1)\right)\Big|_{x=1} = 2.1(1^2-1+1) + (1^2+1)(2.1-1)$$

$$= 2+2 = 4.$$

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Ex. Find $\frac{d}{dx}\left(\frac{x^2+x^3}{\sqrt{x}}\right)$

Sol.ⁿ. $\frac{d}{dx}\left(\frac{x^2+x^3}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{\frac{3}{2}} + x^{\frac{5}{2}}\right)$

$$= (x^{\frac{3}{2}})' + (x^{\frac{5}{2}})'$$

$$= \frac{3}{2}x^{\frac{1}{2}} + \frac{5}{2}x^{\frac{3}{2}}.$$

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