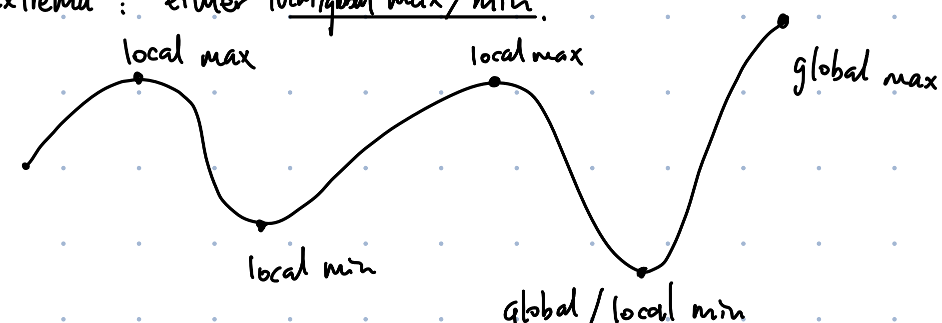
(*) Optimization (§ 10.1, 10.2)

- Motivation: we want to understand where we can find the maximum/minimum of a function.

Def.ⁿ Let x_0 be a point in the domain of $f(x)$.

- x_0 is a <u>local maximum point</u> of $f(x)$, if $f(x_0) \geq f(x)$ for all x near x_0. We say $f(x_0)$ is a <u>local maximum</u> for $f(x)$.
- x_0 is a <u>global maximum point</u> of $f(x)$, if $f(x_0) \geq f(x)$ for all x in the domain of f. We say $f(x_0)$ is a <u>global maximum</u> for $f(x)$.
- We define the <u>local minimum</u> / <u>global minimum</u> by replacing $\geq$ with $\leq$ in above.
- Extrema: either <u>local/global max/min</u>.



- The local max/min only make sense at <u>NOT</u> endpoints.

Q: How to find the local min/max of $f(x)$?

- The local min/max may only occur where $f'(x_0) = 0$. Those points are called stationary points.
- <u>NOT</u> every stationary point is a local min/max point!
- Distinguish local min/max if $f'(x_0) = 0$:

$\boxed{\cup}$ First derivative test: f' transitions from $-$ to $+$ past x_0
Second derivative test: $f''(x_0) > 0 \Leftrightarrow$ concave up
$\Big\}\Rightarrow$ local min

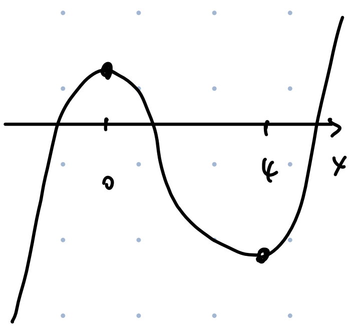
$\boxed{\cap}$ First derivative test: f' transitions from $+$ to $-$ past x_0
Second derivative test: $f''(x_0) < 0 \Leftrightarrow$ concave down
$\Big\}\Rightarrow$ local max.

$\boxed{\lrcorner}$ First derivative test: f' transitions from $+$ to $+$ or $-$ to $-$ past x_0.
$\Rightarrow$ <u>NOT</u> local max/min.

Second derivative test: $f''(x_0) = 0 \Rightarrow$ <u>inflection point</u>: No conclusion.

Ex. Find all stationary points of $f(x) = x^3 - 6x^2 + 1$ on $(-\infty, \infty)$. Are they local max or min?

Sol^n $f'(x) = 3x^2 - 12x = 0$, $3x(x-4) = 0$, $x = 0$ or $x = 4$.

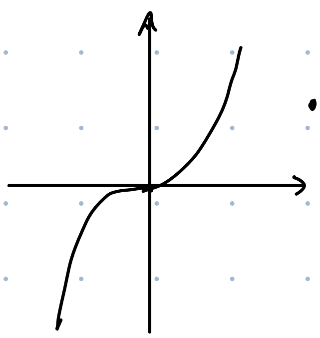


$f''(x) = 6x - 12$

- $x = 0$: $f''(0) = -12 < 0$, concave down $\cap$
  $\Rightarrow$ local max at $x = 0$. (Second derivative test)
- $x = 4$: $f''(4) = 6(4) - 12 = 12 > 0$, concave up $\cup$
  $\Rightarrow$ local min at $x = 4$. (Second derivative test) $\boxed{4}$

Ex. Find all stationary points of $f(x) = x^3$. Are they local max/min?

Sol^n $f'(x) = 3x^2 = 0$, $x = 0$.



$f''(x) = 6x$.

- At $x = 0$, $f''(0) = 0$ $\Rightarrow$ inflection point: no conclusion. (Second der. test)
- Have to use first der. test:

$$\left.\begin{aligned} f'(x) &= 3x^2 > 0 \text{ for } x < 0 \\ f'(x) &= 3x^2 > 0 \text{ for } x > 0 \end{aligned}\right\} \Rightarrow f' \text{ transitions from + to +}$$

$\Rightarrow$ <u>NOT</u> local max/min.
(first der. test)

<u>Optimization:</u> How do we find the global max/min of a function $f(x)$?

<u>Def^n</u>. We say x_0 is a <u>critical point</u> of $f(x)$, if one of the following holds:

1. $f'(x_0) = 0$ (Stationary point)
2. $f'(x_0)$ is undefined.
3. x_0 is an endpoint of the domain of f.

Strategy of optimization:
1. Find all critical points.
2. Compute $f(x)$ at the critical points.
3. Global max = The largest value among those.
4. Global min = the smallest value among those.

Ex. Find the global max/min of $f(x) = x^3 - 3x$, on domain $[-1, 3]$

Sol.^{n} • Crit points: ① $x=-1$, ② $x=3$.

$f'(x) = 3x^2 - 3 = 0$, ③ $x = 1$ ④ $x = -1$

• f at crit points: ① $f(-1) = 2$, ② $f(3) = 18$, ③ $f(1) = -2$, ④ $f(-1) = 2$.

• Global max: $f(3) = 18$.

• Global min: $f(1) = -2$.

Ex. • Find the global max/min of x^4 on $(-\infty, \infty)$.

• Can an inflection point be a local max/min?

Sol.^{n} • $\pm\infty$ - behaviour: ① $x \to \infty$, ② $x \to -\infty$.

• Crit point: $f'(x) = 4x^3 = 0$, ③ $x = 0$.

• f at crit point: ③ $f(0) = 0$

• f at $\pm\infty$: ① $\lim_{x\to\infty} f(x) = \infty$, ② $\lim_{x\to\infty} f(x) = \infty$.

• Global max: don't exist.

• Global min: $f(0) = 0$.

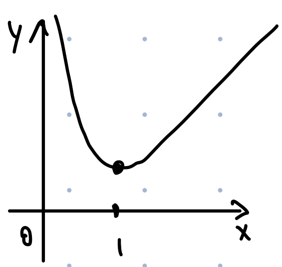
• $f''(x) = 12x^2$. $f''(0) = 0$ $\overset{\text{(second der. test)}}{\Rightarrow}$ 0 is an inflection point: no conclusion.

• $\left.\begin{array}{ll} f'(x) = 4x^3 < 0 & \text{for } x < 0 \\ f'(x) = 4x^3 > 0 & \text{for } x > 0 \end{array}\right\} \Rightarrow f'$ transition from $-$ to $+$ " $\smile$ "

(first der. test) $\Rightarrow$ $f(0) = 0$ is a local min.

• An inflection pt can be a local min.

Ex. Find the global max/min of $f = x + \frac{1}{x}$ on $(0, \infty)$.



Sol.^{n} • As ① $x \to 0^+$, ① $f(x)$ = small + large $\to \infty$.

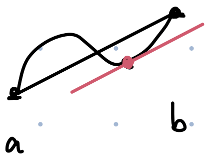
• As ② $x \to \infty$, ② $f(x)$ = large + small $\to \infty$.

• Critical points: $f'(x) = 1 - x^{-2} = 0$, ③ $x = 1$.

• f at crit point: ③ $f(1) = 1 + \frac{1}{1} = 2$.

• Global max: don't exist. • Global min: $f(1) = 2$.

Thm (Mean Value theorem, Appendix E). Let $f(x)$ be continuous on $[a,b]$ and be differentiable on (a,b). Then there is $c \in (a,b)$ that



$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$