(*) Inverse functions (§12.1 — 12.3)

- Motivation: the inverse function of $f(x)$ undo $f(x)$.

Def^n. We say $g(x)$ is the inverse function of $f(x)$, if
$$\begin{cases} g(f(x)) = x & \text{for all } x \text{ in the domain of } f. \\ f(g(x)) = x & \text{for all } x \text{ in the domain of } g. \end{cases}$$

- If $g(x)$ is the inverse of $f(x)$, $f(x)$ is also the inverse function of $g(x)$. We denote $g(x)$ by $f^{-1}(x)$.

Def^n. We say $f(x)$ is invertible if $f(x)$ has an inverse.

- $f(x)$ is invertible if and only if $f(x)$ is 1-1, that is, for every y in the range of f, there is exactly 1 element x in the domain of f that $f(x) = y$.

Ex. Verify that $\begin{cases} f(x) = x^{\frac{1}{4}} & \text{on } (0,\infty) \\ g(x) = x^4 & \text{on } (0,\infty) \end{cases}$ are inverse functions.

Sol^n. • For each x in $(0,\infty)$, $(g\circ f)(x) = g(f(x)) = g(x^{\frac{1}{4}}) = (x^{\frac{1}{4}})^4 = x$.
• For each x in $(0,\infty)$, $(f\circ g)(x) = f(g(x)) = f(x^4) = (x^4)^{\frac{1}{4}} = x$.
So they are inverse functions. $\boxed{\text{Ez}}$

- Strategy in finding the inverse function:
  1. Write x using y.
  2. Switch x and y in the function.

Ex. Find the inverse function of $f(x) = x^3$

Sol^n ① $y = x^3$, $y^{\frac{1}{3}} = (x^3)^{\frac{1}{3}} = x$, $x = y^{\frac{1}{3}}$.
② $x = y^{\frac{1}{3}} \xrightarrow{\text{switch}} y = x^{\frac{1}{3}}$.
So $f^{-1}(x) = x^{\frac{1}{3}}$.

Ex. Find the inverse function of $f(x) = 4x^5 + 9$.

Sol^n. ① $y = 4x^5 + 9$, $y - 9 = 4x^5$, $\frac{y-9}{4} = x^5$, $x = \left(\frac{y-9}{4}\right)^{\frac{1}{5}}$.
② Switch: $y = \left(\frac{x-9}{4}\right)^{\frac{1}{5}}$.

So the inverse function is $f^{-1}(x) = \left(\frac{x-9}{4}\right)^{\frac{1}{5}}$.

Q. Is $f(x) = x^2$ invertible on $(-\infty, \infty)$?

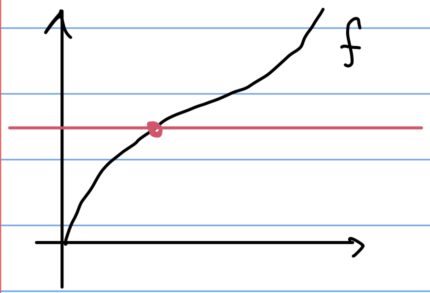
Attempt: $y = x^2$, $\sqrt{y} = \sqrt{x^2} \overset{?}{=} x$.

But x may be negative on $(-\infty,\infty)$, whence $\sqrt{x^2} = -x$.

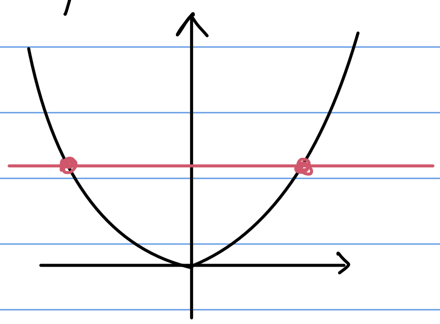
Indeed, $f(-1) = f(1) = 1$, so f cannot be 1-1.

- Horizontal line test: $f(x)$ is invertible if and only if every horizontal line intersects the graph of $f(x)$ at exactly one point.

Ex.



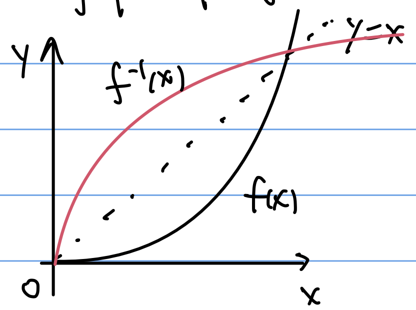
- f invertible from $[0,\infty)$ to $[0,\infty)$.



- f not invertible from $(-\infty,\infty)$ to $[0,\infty)$
- But f invertible from $[0,\infty)$ to $[0,\infty)$.

- The graph of $f(x)$ and $f^{-1}(x)$ are symmetric about $y=x$.

Ex.



- If $f(x)$ keeps increasing ($f(x) > f(y)$ when $x > y$) on its domain, then f is invertible. Similarly true for $f(x)$ keeps decreasing.
- Then if differentiable $f(x)$ has no stationary points, $f(x)$ is invertible.

Ex. Show $f(x) = x^3 + 3x^2 + 10x - 20$ is invertible on $(-\infty, \infty)$

Sol.: $f'(x) = 3x^2 + 6x + 10 = 3(x^2 + 2x + 1) + 7 = 3(x+1)^2 + 7 > 0$ everywhere.

• $f(x)$ keeps increasing on $(-\infty, \infty)$.

So $f(x)$ is 1-1 and hence invertible.

Ex. Let $f(t)$ = temperature (°F) of pizza in the fridge t mins after refridgerating.

Assume $f(t)$ keeps decreasing, and is hence invertible.

Then $f^{-1}(32)$ = amount of time needed for the pizza to cool down to 32 °F.