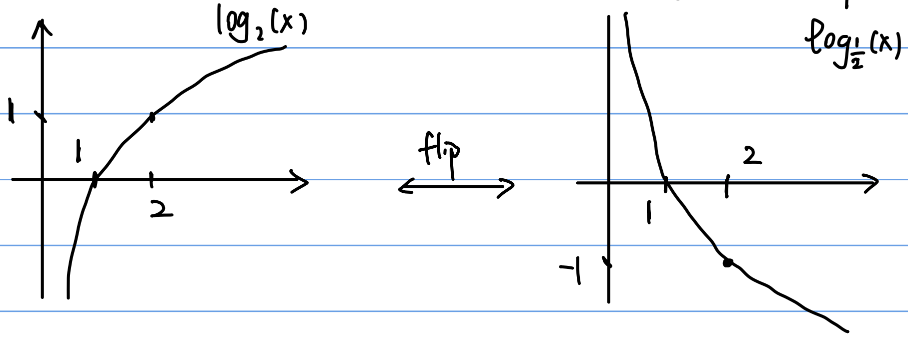
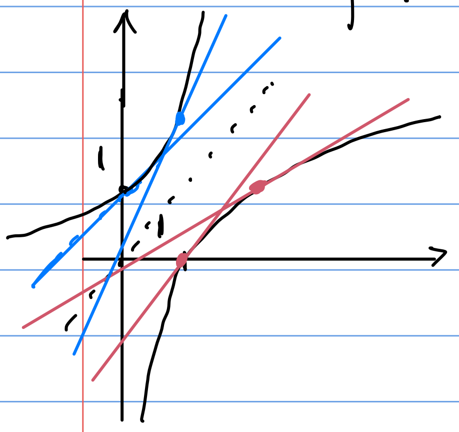
- More facts about log's:
- $\log_b(1/b) = \log_b(1) - \log_b(b) = 0 - 1 = -1.$
- $\log_{1/b}(x) = \frac{\log_b(x)}{\log_b(1/b)} = -\log_b(x).$
- So $\log_{1/b}(x)$ has the graph of $\log_b(x)$ fliped vertically:

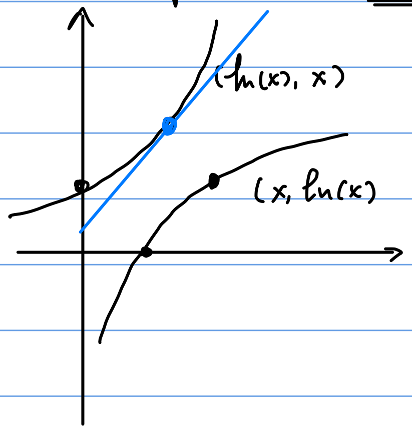


⊛ Derivatives of logarithms and exponentials.

- Goal of this lecture: know how to compute the derivatives of log's and exp's (we already know this.)

- Find the tangent line to the graph of ln(x) ?





tangent line to e^x at ln(x) : slope $= \frac{d}{dx}e^x\Big|_{x=\ln(x)} = e^x\Big|_{x=\ln(x)}$
$= e^{\ln x} = x.$

tangent line to ln(x) at x must has reciprocal slope $= \frac{1}{x}$.

Thm $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$.

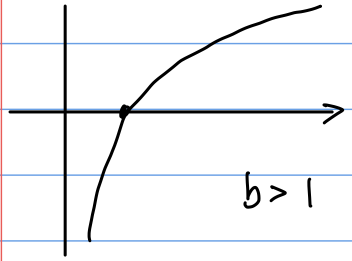
- How about $\frac{d}{dx}\log_b(x)$?

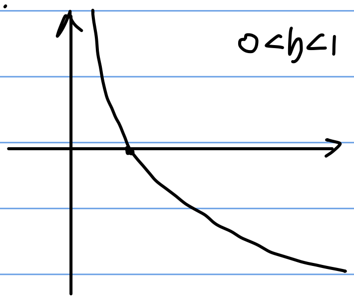
Thm Let $b>0$ and $b\neq 1$. Then $\frac{d}{dx}\log_b(x) = \frac{1}{\ln(b)}\cdot\frac{1}{x}$.

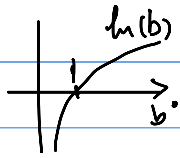
Pf. Note via the change of base, $\log_b(x) = \frac{\ln(x)}{\ln(b)}$, and $\ln(b)$ is a fixed constant. So

$$\frac{d}{dx}(\log_b(x)) = \frac{d}{dx}\left(\frac{1}{\ln(b)}\ln(x)\right) = \frac{1}{\ln(b)}\frac{d}{dx}\ln(x) = \frac{1}{\ln(b)}\frac{1}{x}. \ \square$$

- Concavity of $f(x)=\log_b(x)$:



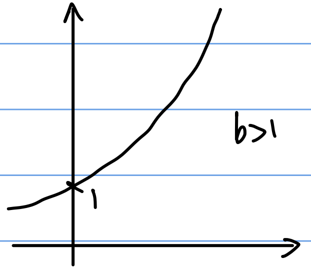


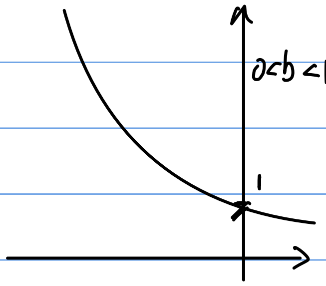


- When $b>1$, $f'(x) = \frac{1}{\ln b}\frac{1}{x}$, $f''(x) = -\frac{1}{\ln b}x^{-2} < 0$,
  since $\ln b > 0$ as $b>1$. ⇒ concave down
- When $0<b<1$, $f''(x) = -\frac{1}{\ln b}x^{-2} > 0$,
  since $\ln b < 0$ as $0<b<1$ ⇒ concave up

(known)
Thm $\frac{d}{dx}b^x = (\ln b)\,b^x$.

- Concavity of $f(x)=b^x$:





- both cases: $f'(x) = (\ln b)\,b^x$, $f''(x) = (\ln b)(\ln b)\,b^x = (\ln b)^2 b^x > 0$,
  concave up .

- Examples on practically compute derivatives of b^x and $\log_b x$:

Ex Find $\frac{d}{dx}\ln\left(\frac{x^5}{2}\right)$.

Sol.ⁿ Firstly simplify:

$$f(x)=\ln\left(\frac{x^5}{2}\right) = \ln(x^5) - \ln(1)$$
$$= 5\ln(x) - \ln(2).$$

- So $\frac{d}{dx}f(x) = 5\frac{d}{dx}(\ln(x)) = 5/x$.

Ex Find $\frac{d}{dx}(x^2 2^x)\big|_{x=1}$

Sol.ⁿ · $\frac{d}{dx}(x^2 2^x) = \frac{d}{dx}(x^2)\,2^x + x^2\frac{d}{dx}(2^x)$

$$= 2x\,2^x + x^2(\ln 2)(2^x)$$

- $\frac{d}{dx}(x^2 2^x)\big|_{x=1} = 2(1)2^1 + 1^2\ln(2)(2^1) = 4 + 2\ln(2)$

Ex. Find $\frac{d}{dx}(7^{5x+2})\big|_{x=0}$

Sol.ⁿ Simplify: $f(x)=7^{5x+2} = 7^2\,7^{5x} = 49\,(7^5)^x$.

- $\frac{d}{dx}f(x) = \frac{d}{dx}(49(7^5)^x) = 49\ln(7^5)(7^5)^x$. 4
- $\frac{d}{dx}f(x)\big|_{x=0} = 49\ln(7^5)(7^5)^0 = 49(5)\ln(7) = 245\ln(7)$.

Ex Find $\frac{d}{dx}\frac{\overbrace{\ln(x^2)}^{f}}{\underbrace{e^{3x}}_{g}}$

Sol.ⁿ $\begin{cases} f(x) = 2\ln(x), & f'(x) = \frac{2}{x} \\ g(x) = e^{3x}, & g'(x) = (e^3)^x = \ln(e^3)\,e^{3x} = 3e^{3x}. \end{cases}$

- So $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} = \frac{\frac{2}{x}e^{3x} - 2\ln(x)(3e^{3x})}{(e^{3x})^2} = \frac{\frac{2}{x}e^{3x} - 6\ln(x)e^{3x}}{e^{6x}}$

$$= \frac{\frac{2}{x} - 6\ln(x)}{e^{3x}}$$

Ex. Find $\frac{d}{dx}\ln\left(\frac{1}{\sqrt{x}}\right)$.

Sol.ⁿ Simplify: $f = \ln\left(\frac{1}{\sqrt{x}}\right) = \ln\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}\ln(x)$.

So $\frac{d}{dx}\ln\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(-\frac{1}{2}\ln(x)\right) = -\frac{1}{2}\frac{1}{x} = -\frac{1}{2x}$.

Ex. Find $\frac{d}{dx} \log_x(e)$.

Sol^n. Simplify : $\log_x(e) \overset{\text{change of base}}{=} \frac{\ln(e)}{\ln(x)} = \frac{1}{\ln(x)} \begin{array}{l} - f \\ - g \end{array}$.

$f = 1, \quad f' = 0,$

$g = \ln(x), \quad g' = \frac{1}{x}.$

$\cdot \ \frac{d}{dx} \log_x(e) = \frac{f'g - fg'}{g^2} = \frac{-\frac{1}{x}}{(\ln(x))^2} = -\frac{1}{x(\ln(x))^2}$.