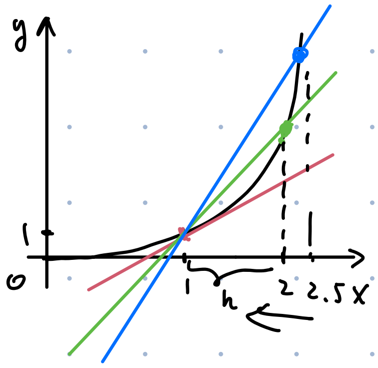
(*) Derivatives (§ 5.1, 5.2)

Ex. Find the slope of the "tangent line" to $f(x)=x^2$ at $x=1$.



- tangent line = the best approximation line to $(\overset{x}{1}, \overset{y}{1})$.
- Strategy: approximate the tangent line via secant lines.

  Blue $\xrightarrow{\text{closer}}$ Green $\xrightarrow{\text{closer}}$ ... $\xrightarrow{\text{approximate}}$ tangent line.
- Secant line through $(1,1)$ to $(1+h, (1+h)^2)$ has the form

  $y = m(x-1)+1$, $m = \underline{ARC} = \frac{\Delta y}{\Delta x} = \frac{(1+h)^2 - 1^2}{1+h-1} = 2+h$.

- $h=1$, $m=3$,
- $h=0.1$, $m=2.1$
- $h=0.01$, $m=2.01$
- $h=0.0001$, $m=2.0001$

$\Big\}$ As $h\to 0$, the approximating secant lines has a slope closer and closer to 2.

So, the tangent line should be $y=2(x-1)+1$.

- On another hand, we saw the $\underline{ARC}$ through $(1,1)$ and nearby points on the curve gets closer to 2 as the points get closer to $(1,1)$.
- We call this "limit" of average rate of change of f, the <u>instaneous rate of change of f</u> at $x=1$, $\underline{f'(1)}$, the <u>derivative</u> of f at 1.

- Informal definition of "limit": for a function $g(h)$, we say

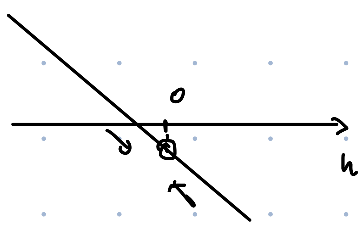
  As $h\to 0$, $g(h) \to$ constant c, or say, $\lim_{h\to 0} g(h) = c$,

  if $g(h)$ is close to c whenever $h\neq 0$ gets small.

Ex: In earlier example, $m=m(h)$ gets closer to 1 as $h\neq 0$ gets small, so $\lim_{h\to 0} m(h) = 1$.

Ex: For $f(h)=h-1$, $f(1)=0$, $f(0.1)=-0.9$, $f(0.01)=-0.99$,

$f(h)$ gets closer to -1 as $h\neq 0$ gets small, so $\lim_{h\to 0} f(h) = -1$.



Def^n. Let c be in the domain of f. We denote $f'(c)$ by
$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0}\left(\frac{\Delta f}{\Delta x} \text{ over } [c, c+x]\right),$$
if the "limit" exists. We call $f'(c)$, <u>the derivative</u> of f at $x=c$.

- This is the limit definition of derivative.
- $f'(c)$ = the slope of tangent line to f at c = the instaneous rate of change of f at c.

- We say $f(x)$ is <u>differentiable</u> at c if $f'(c)$ exists.
- We say $f(x)$ is <u>differentiable</u> on interval (a,b) if $f'(c)$ exists for each $c\in(a,b)$.

<u>Ex</u> Find the derivative of $f(x)=x^2$ at $x=c$.

<u>Sol^n</u>. $f'(c) = \lim_{h\to 0}\left(\frac{\Delta f}{\Delta x} \text{ over } (c, c+h)\right) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$
$$= \lim_{h\to 0}\frac{(c+h)^2-c^2}{h} = \lim_{h\to 0}\frac{c^2+2hc+h^2-c^2}{h} = \lim_{h\to 0}(2c+h).$$

When $h\neq 0$ is small, $2c+h = 2c + \text{small} \approx 2c$, so
$$f'(c) = 2c.$$

- This implies $f'(1)=2$, consistent with out first example.

<u>Ex</u>. Find the tangent line of $f(x) = 1/x^2$ through $(1,1)$.

<u>Sol^n</u> • We need to know the slope of tangent line ($= f'(1)$).

- Compute the derivative: $(c\neq 0)$
$$f'(c) = \lim_{h\to 0}\left(\frac{\Delta f}{\Delta x} \text{ over } (c, c+h)\right) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{1}{(c+h)^2}-\frac{1}{c^2}}{h} = \lim_{h\to 0}\frac{c^2-(c+h)^2}{c^2(c+h)^2 h}$$
$$= \lim_{h\to 0}\frac{-2ch-h^2}{c^2(c^2+h^2)h} = \lim_{h\to 0}\frac{-2c-h}{c^2(c^2+h^2)}$$

- For $h\neq 0$ small,
$$-2c-h \approx -2c, \quad c^2(c^2+h^2)\approx c^4.$$

- $f'(c) = \frac{-2c}{c^4} = -\frac{2}{c^3}$. • $f'(1) = -2$

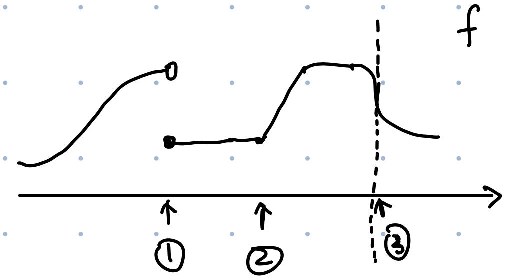
- Tangent line: $y = -2(x-1)+1 = -2x+3$.

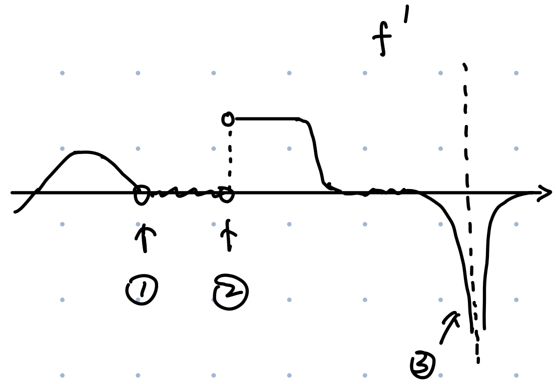
• The derivative $f'(c)$ is not defined if any of the following:

① f is not continuous at c.

② f has sharp corner at c.

③ the tangent line of f at $x=c$ is vertical. ($f'(c)$ doesn't exist as a limit.)





Def^n. Let $f(x)$ be a function on (a,b) with $f'(c)$ exists for each $c \in [a,b]$.

Then

input	output
c	$\to f'(c)$

is a function on $c \in (a,b)$.

We define the derivative function $f'(x)$ of $f(x)$ by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

• We transformed a function $f(x)$ into another function $f'(x)$.

• $f'(x)$ = slope function of $f(x)$ at x.

Ex: Find the derivative of $f(x)=\sqrt{x}$, on $(0,\infty)$.

Sol^n.
$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$(A-B)(A+B) = A^2-B^2$.

$$= \lim_{h\to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

Ex: Find the derivative of $f(x)=\frac{1}{x^2}$. Is it defined everywhere?

Sol^n
$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

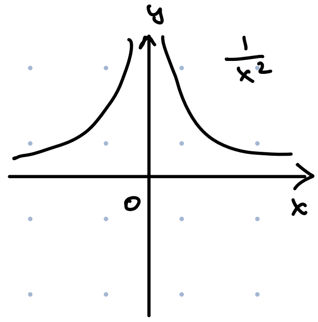
numerator $= \frac{x^2}{(x+h)^2x^2} - \frac{(x+h)^2}{x^2(x+h)^2} = \frac{x^2-(x^2+2hx+h^2)}{x^2(x+h)^2} = \frac{-2hx+h^2}{x^2(x+h)^2}$.

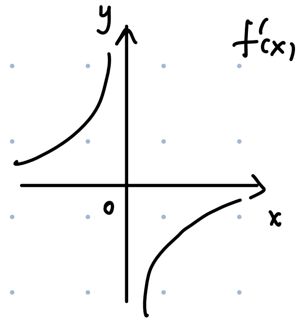
$$f'(x) = \lim_{h\to 0} \frac{-2x+h}{x^2(x+h)^2}.$$

As $h\neq 0$ small, $-2x+h$ is close to $-2x$, $x^2(x+h)^2$ is close to x^4.

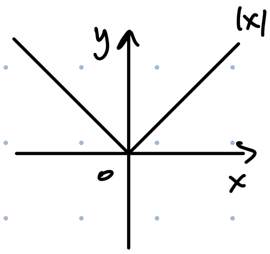
So $f'(x) = \frac{-2x}{x^4} = -\frac{2}{x^3}$.

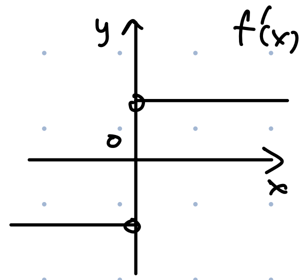
$f'(x)$ is defined on $\{x \neq 0\}$.





Ex. Find the derivative of $f(x) = |x|$.





① When $x > 0$, $f(x) = x$,

$$f'(x) = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

② When $x < 0$, $f(x) = -x$

$$f'(x) = \lim_{h \to 0} \frac{-(x+h)-(-x)}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

③ When $x = 0$, $f(0+h) = \begin{cases} h & \text{if } h > 0 \\ -h & \text{if } h < 0. \end{cases}$, $f'(0) = \lim_{h \to 0} \frac{|h|}{h}$

For h small positive, $\frac{|h|}{h} = 1$,

For h small negative, $\frac{|h|}{h} = \frac{-h}{h} = -1$.

So $\frac{|h|}{h}$ is not close to 1 nor -1.

The limit does not exist.