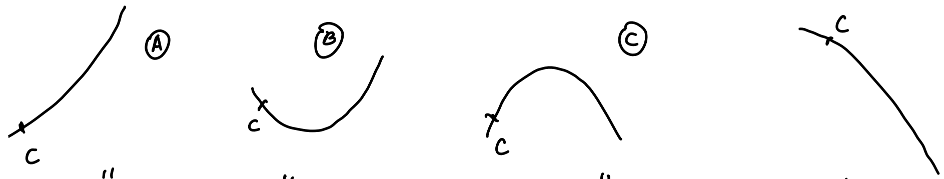
(*) Interpretation of the derivative.

- $f'(x) = 5$ means, if x is increased by 1 unit, f would "roughly" increase by 5 units.

- $f'(c) > 0 \iff f$ is increasing near $x = c$.
- $f'(c) < 0 \iff$ ······ decreasing ·······.
- $f'(c) = 0 \iff f$ is roughly the same ······.



"concave up" "concave down"

Ⓐ $f'(c) > 0$ and f' keeps increasing : f is increasing faster and faster.
Ⓑ $f'(c) < 0$ and f' keeps increasing : f is decreasing, but try to increase gradually.
Ⓐ, Ⓑ : $f'(x)$ increase = "concave up near c".

Ⓒ $f'(c) > 0$ and f' keeps decreasing : f is increasing, but try to decrease.
Ⓓ $f'(c) < 0$ and f' keeps decreasing : f is decreasing faster and faster.
Ⓒ, Ⓓ : $f'(x)$ decrease = "concave down near c".

Ex. The accumulated profit function $R(x)$ of a company is modelled by

$$R(x) = \frac{1}{2}(x-500)^2 + 2x,$$

for $0 \le x \le 1500$, where x is the number of goods produced.

① Find the derivative $R'(x)$ on $0 < x < 1500$.

② Plot the graphes of $R'(x)$.

③ Interpret $R'(x)$ with-in the context.

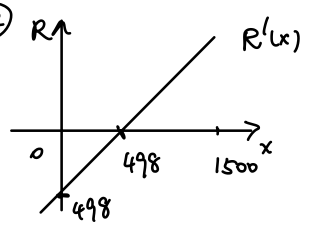
Sol^h ① $R'(x) = \lim_{h\to 0} \frac{\Delta R}{\Delta x} = \lim_{h\to 0} \frac{R(x+h) - R(x)}{h}$.

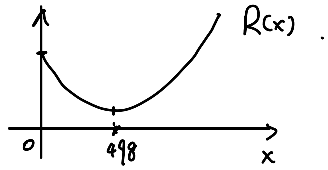
$R(x+h) = \frac{1}{2}(x+h-500)^2 + 2(x+h) = \frac{1}{2}\left((x-500)^2 + 2h(x-500) + h^2\right) + 2x + 2h$

$R(x) = \frac{1}{2}(x-500)^2 + 2x$.

$R(x+h) - R(x) = h(x-500) + \frac{1}{2}h^2 + 2h = h(x-498) + \frac{1}{2}h^2$.

$$R'(x) = \lim_{h \to 0} \frac{h(x-498) + \frac{1}{2}h^2}{h} = \lim_{h \to 0} x - 498 + \frac{1}{2}h = x - 498 \quad \text{on} \quad 0 < x < 1500.$$

② 



③ $R'(x)$ = the change in profit if increase production x by 1

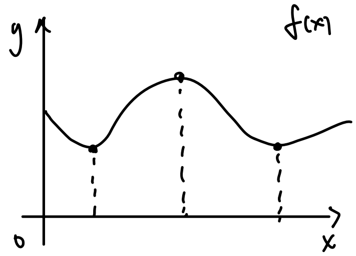
= marginal profit of product.

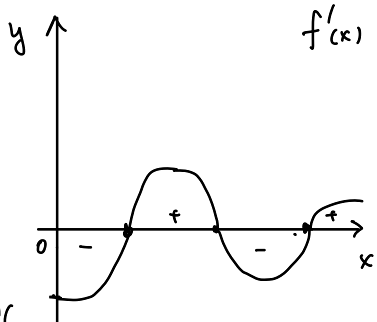
- $R'(x) < 0$ on $x \in (0, 498)$: for the first 498 products, the company are losing money.
- $R'(x) > 0$ on $x \in (498, 1500)$: the company is making more and more money for each product produced after # 498. ☒

<u>Remarks</u>: Key observations of $f'(x)$:

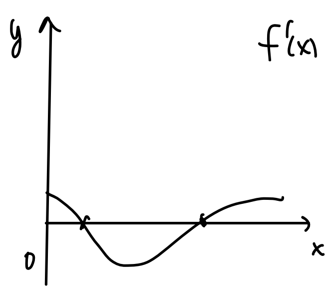
- The zeroes of $f'(x)$: f is not changing locally (horizontal tangent line)
- If $f'(x)$ transitions from $-$ to $+$: f started decreasing for a while but transitioned into increasing.
- If $f'(x)$ is undefined: f has a sharp corner or a jump.

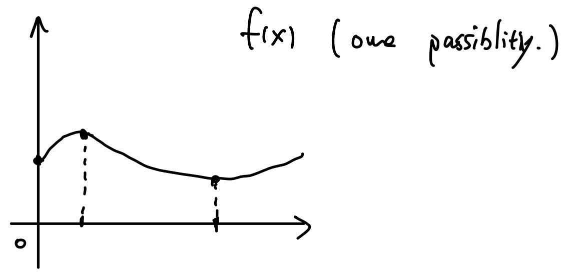
<u>Ex</u>: Roughly plot $f'(x)$ based on $f(x)$:



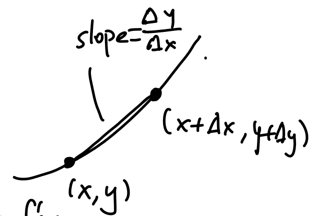


<u>Ex</u>: Roughly plot $f(x)$ based on $f'(x)$.





⊛ Derivative : more on notations (S 5.4)



- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \underset{\text{denoted}}{=\!=} \frac{dy}{dx}$.

  ↑ not a fraction but a notation for $f'(x)$.

- Equivalent notation for $f'(x)$, as a function

  f', $f'(x)$, y', $\frac{dy}{dx}$, $\frac{df}{dx}$.

- Equivalent notation for f' at $x=c$, as a value:

  $f'(c)$, $y'(c)$, $\frac{dy}{dx}\Big|_{x=c}$, $\frac{df}{dx}\Big|_{x=c}$.

- $\frac{d}{dx}$: take the derivative of what follows after, as a function:

  $\frac{d}{dx}(f) = \frac{df}{dx}$, $\frac{d}{dx}(x^2) = 2x$.

Ex: Find $\frac{d}{dx}(x^3+4x)$ and $\frac{d}{dx}(x^3+4x)\Big|_{x=2}$:

Sol^n: $\frac{d}{dx}(x^3+4x) = \frac{df}{dx}$, $f(x) = x^3+4x$.

$$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3+4(x+h)-(x^3+4x)}{h}$$

- numerator $= (x+h)(x^2+2hx+h^2) - x^3 - \cancel{4x} \; \cancel{+4x} + 4h = \cancel{x^3} + 2hx^2 + h^2x + hx^2 + 2h^2x + h^3 \; \cancel{-x^3} + 4h$

  $= 3hx^3 + 3h^2x + h^3$.

$$\frac{df}{dx} = \lim_{h\to 0}\frac{3hx^2+3h^2x+h^3+4h}{h} = \lim_{h\to 0} 3x^3+4 + \underbrace{3hx+h^2}_{\text{small}} = 3x^2+4.$$

- So $\frac{d}{dx}(x^3+4x) = 3x^2+4$.
- $\frac{d}{dx}(x^3+4x)\Big|_{x=2} = f'(2) = 3\times 2^2+4 = 16$.