(*) Algebra of functions (§ 3.1, 3.2, 3.3)

- Addition: $(f+g)(x) = f(x) + g(x)$, for any x in both domains of f and g.
- Subtraction: $(f-g)(x) = f(x) - g(x)$, ———— " ————.
- Multiplication: $(fg)(x) = f(x)g(x)$, ———— " ————.
- Division: $(f/g)(x) = \dfrac{f(x)}{g(x)}$, for any x in both domains of f and g, and $g(x) \neq 0$.

Ex. Find the domain of $\dfrac{(x+3)^2}{\sqrt{x} - x}$.

Domain$(\sqrt{x}) = [0, \infty)$, Domain$(x) = (-\infty, \infty)$,

Domain$(\sqrt{x} - x) = [0, \infty)$, Domain$((x+3)^2) = (-\infty, \infty)$,

$\sqrt{x} - x = 0$ if $\sqrt{x} = x$, that is $\sqrt{x} = 1$, $x = 1$.

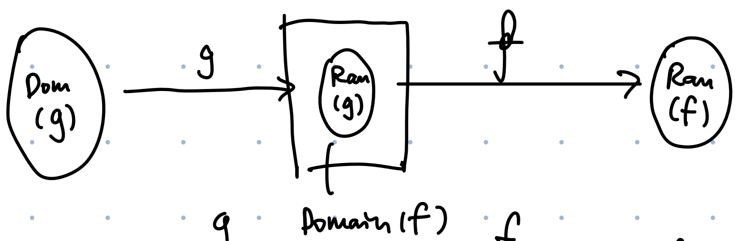
So Domain$\left(\dfrac{(x+3)^2}{\sqrt{x} - x}\right) = \{x \geq 0, x \neq 1\}$.

- Composition of functions: the composite function of f, g is defined as

$$(f \circ g)(x) = f(g(x)),$$

whose domain is the set of all x in the domain of g, and $g(x)$ is in the domain of f.

- Order of f, g matters!
- Closer to input means compute first: $f \circ g(x)$ means first do g, then do f.



$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x)).$$

Ex. Let $f(x) = x^3$, $g(x) = \dfrac{1}{x+8}$. Find $f \circ g$ and $g \circ f$.

Sol^n $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x+8}\right) = \left(\dfrac{1}{x+8}\right)^3 = \dfrac{1}{(x+8)^3}$, domain $= \{x \neq -8\}$.

$(g \circ f)(x) = g(f(x)) = g(x^3) = \dfrac{1}{x^3 + 8}$, domain $= \{x \neq -2\}$.

- It's safer to follow the rule to compute the domain!
- In general, $g \circ f \neq f \circ g$.

Ex. Write $\sqrt{x^2+7}$ as the composition of three functions ($\sqrt{x}$, x^2, addition).

Sol^n. Read it off: square, then add 7, then square root.

Let $h(x)=x^2$, $g(x)=x+7$, $f(x)=\sqrt{x}$.

Then $\sqrt{x^2+7} = f(g(h(x))) = (f\circ g\circ h)(x)$.

Ex. Let $f(x)=x^2$, $g(x)=\sqrt{x}$. Find $g\circ f$, $f\circ g$ and their domains and ranges

Sol^n. • $(g\circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$.

• $(f\circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$.

• Domain$(f) = (-\infty,\infty)$, Domain$(g) = [0,\infty)$.

$\begin{cases} \text{Domain}(g\circ f) = \text{all } x \text{ that } f(x) \text{ is in domain}(g) = (-\infty,\infty) \\ \text{Range}(g\circ f) = (-\infty,\infty) \end{cases}$

$\begin{cases} \text{Domain}(f\circ g) = \text{all } x \text{ that } g(x) \text{ is in domain}(f) = [0,\infty) \\ \text{Range}(f\circ g) = [0,\infty) \end{cases}$

$\boxed{\text{Ex}}$

• Informally, we say g is the inverse function of f, if

$\begin{cases} f\circ g(x) = x & \text{for each } x\in \text{Domain}(f\circ g). \\ g\circ f(x) = x & \text{for each } x\in \text{Domain}(g\circ f). \end{cases}$

Ex. Find the inverse of $f(x)=x^3$.

Sol^n. Let $y=f(x)=x^3$, so $y^{\frac{1}{3}}=x$.

The inverse function is $g(y)=y^{\frac{1}{3}}$. (A.K.A. $g(x)=x^{\frac{1}{3}}$)

Check: $(f\circ g)(x) = f(g(x)) = f(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x$.

$(g\circ f)(x) = g(f(x)) = g(x^3) = (x^3)^{\frac{1}{3}} = x$. ✓.

• Find the inverse: first write $y=f(x)$, then write x in term of y.

Ex. Let $f(x)=(x+5)^2-9$, $g(x)=\frac{1}{1+x}$. Find the zeroes of $f\circ g(x)$.

Sol^n $f\circ g(x) = f(g(x))$.

• When if $f(z)=0$? $f(z)=(z+5)^2-9=0$, if $(z+5)^2=9$, if $z=-2,-8$.

- $f(g(x)) = 0$ if $g(x) = -2$ or -8.

$$\begin{cases} g(x) = \frac{1}{1+x} = -2 & \text{if } x = -\frac{3}{2}. \\ g(x) = \frac{1}{1+x} = -8 & \text{if } x = -\frac{9}{8}. \end{cases}$$

So at $x = -\frac{3}{2}$ or $-\frac{9}{8}$, $(f \circ g)(x) = 0$.