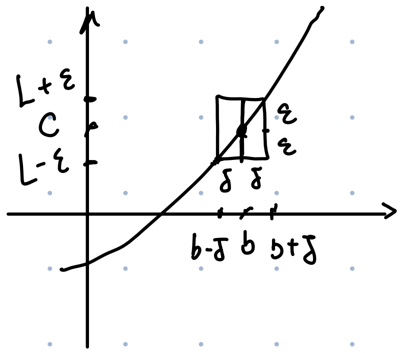
(*) Limits, revisited. (§7.1, §7.2, §7.3).

- $\lim_{h\to 0} f(h) = L$ means "$f(h)$ is close to L, when h is small."
- $\lim_{x\to b} f(x) = L$ means "$f(x)$ is close to L, when x is close to b."



- Def.^{n} We say $\lim_{x\to b} f(x) = L$, if for every small $\varepsilon > 0$, there is $\delta > 0$ such that
  $0 < |x-b| < \delta$ guarantees $|f(x) - L| < \varepsilon$.
- We will almost surely never use this definition in this class, since it is very technical. We focus on finding the limit instead.

Ex. Find $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

Sol.^{n} When x is close to 2, $x-2$ is close to 0. Don't want to divide by 0.
Further simplify:

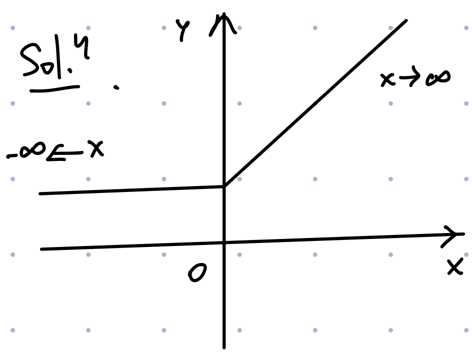
$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2.$$

When x is close to 2, $x+2$ is close to $2+2=4$. So $\lim_{x\to 2} x+2 = 4$.

- Indefinite limits: • We say $\lim_{x\to\infty} f(x) = L$ if as x gets larger and larger, $f(x)$ is closer and closer to L. (Similarly for $\lim_{x\to -\infty} f(x) = L$.)
- We say $\lim_{x\to b} f(x) = \infty$ if as x gets close to b, f gets larger and larger.
- We say $\lim_{x\to b} f(x) = -\infty$ if ________ " ______, f gets smaller and smaller.

Ex. Let $f(x) = \begin{cases} x+1, & x \geq 0 \\ -1, & x < 0. \end{cases}$ Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$.

Sol.^{n}



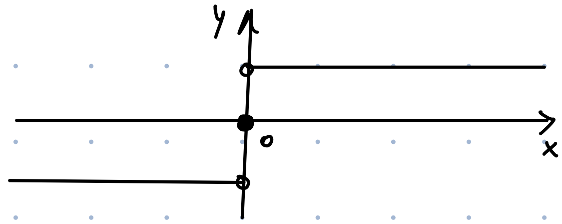
- As $x \to -\infty$, $f(x)$ is close to 1:
  $\lim_{x\to -\infty} f(x) = 1$.
- As $x \to \infty$, $f(x)$ is getting larger and larger.
  So we denote by $\lim_{x\to\infty} f(x) = \infty$. $\boxed{k}$

($\lim = \pm\infty$ means the limit does not exist.)

Def^n. (Left/right limit) • We say $\lim_{x\to b^+} f(x) = L$, if when x is close to b and $x > b$, $f(x)$ is close to L.

• We say $\lim_{x\to b^-} f(x) = L$, if when x is close to b and $x < b$, $f(x)$ is close to L.

<u>Ex</u>. Let $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$.



Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

<u>Sol^n</u> ① As x gets close to 0 with $x > 0$, $f(x) = 1$ is close to 1.
So $\lim_{x\to 0^+} f(x) = 1$.

② As x gets close to 0 with $x < 0$, $f(x) = -1$ is close to -1
So $\lim_{x\to 0^-} f(x) = -1$.

• Limit does <u>NOT</u> depend on the exact value at that point.

• We are now in position to discuss what happens when one divides non-zero number by 0.

<u>Ex</u>. Find the limit $\lim_{x\to 0^+} \frac{1}{x}$, $\lim_{x\to 0^-} \frac{1}{x}$.

<u>Sol^n</u>. ① As x is close to 0 with $x > 0$, $\frac{1}{x}$ is $\frac{1}{\text{small number}}$ = large number, so $\frac{1}{x}$ gets larger and larger. This means $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

② As x is close to 0 with $x < 0$, $\frac{1}{x}$ is $\frac{1}{-\text{small number}}$ = $-$ large number. So $\frac{1}{x}$ gets more negative and negative. So $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. $\boxed{4}$

• We can say $\frac{1}{\text{small number}} \to \infty$ only when <u>NOT</u> both numerator and denominator are small at the same time. ($\lim_{x\to 0^+} \frac{1}{x} = \infty$, but $\lim_{x\to 0^+} \frac{x}{x} = 1$.)

• $\lim_{x\to b} f(x) = L$, if and only if, $\lim_{x\to b^+} f(x) = \lim_{x\to b^-} f(x) = L$.

<u>Ex</u>. Show $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$, does not have a limit at $x = 0$.

<u>Sol^n</u> If $\lim_{x\to 0} f(x)$ exists, then $\underset{\substack{\| \\ -1}}{\lim_{x\to 0^-} f(x)}$ and $\underset{\substack{\| \\ 1}}{\lim_{x\to 0^+} f(x)}$ must exist and agree in value.

But they don't agree. So the limit does not exist.

- How to complete limits? <u>Tricks</u>:

① The real danger is dividing small number by small numbers.

② $\frac{\text{not small}}{\text{small}}$ = large. ($\lim_{x \to 0} \frac{3}{x^2} = \infty$)

③ When $\frac{\text{small}}{\text{small}}$, try the following: (x is close to 1)

$\boxed{\text{I}}$. Factorise: $\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1 \to 2$

$\boxed{\text{II}}$. Rationalise: (use $A - B = \frac{A^2 - B^2}{A+B}$) $\frac{\sqrt{x}-1}{x-1} = \frac{x-1}{(x-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1} \to \frac{1}{2}$.