⊛ Continuity, IVT, EVT. (§7.4)

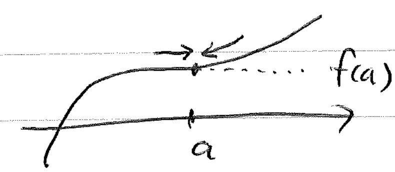
- Continuous function means the function is changing gradually in a traceable way.

Def^n (Continuous) A function is <u>continuous at $x=a$</u> if,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a).$$

This is equivalent to say

$$\lim_{x\to a} f(x) = f(a).$$



- We say f is <u>continuous on an interval</u> (a,b) if f is continuous at every point in that interval.

Ex Show $f(x) = \frac{1}{x}$ is continuous at $x=1$, but NOT continuous at $x=0$.

pf ① $x=1$, $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x}$.

As x is close to 1, $\frac{1}{x}$ is close to $\frac{1}{1}=1$.

So $\lim_{x\to 1} f(x) = 1$, while $f(1) = \frac{1}{1} = 1$.

We have $f(1) = \lim_{x\to 1} f(x) = 1$. Thus $f(x)$ is continuous at $x=1$.

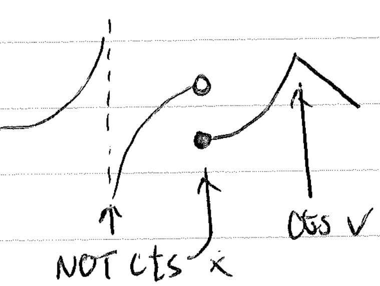
② $x=0$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x}$,

As x is close to 0, $x>0$, $\frac{1}{x} = \frac{1}{\text{small}} = \text{large}$. So

$\lim_{x\to 0^+} f(x) = \infty$, the right limit does not exist,

So f is not continuous at $x=0$.

Ex



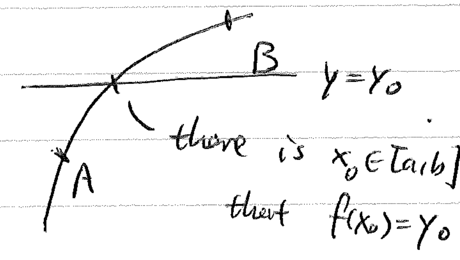
Thm (Intermediate Value Theorem)

Let f be continuous on an closed and bounded interval $[a,b]$ ($a \neq \pm\infty$, $b \neq \pm\infty$), and $f(a) = A$, $f(b) = B$. Then for any number $A \leq y_0 \leq B$, there must be some $x_0 \in [a,b]$ such that

$$f(x_0) = y_0.$$

- **Remark**: Continuous function on $[a,b]$ attains all intermediate values.

Ex There is a continuous function f on $[1,2]$ such that $f(x) \neq 0$ for any $x \in [1,2]$. Can $f(1) < 0$ and $f(2) > 0$?



Sol^{n}. It cannot. Assume $f(1) = A < 0$, $f(2) = B > 0$. Then since 0 is an intermediate value between A and B, there must, from the IVT, some $x_0 \in [1,2]$ that $f(x_0) = 0$. Thus is impossible. $\boxed{a}$

Thm (Extreme Value Theorem)

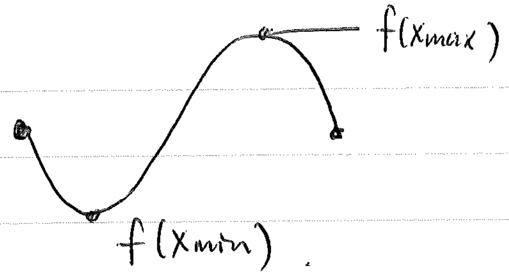
Let f be continuous on a closed and bounded interval $[a,b]$ ($a \neq \pm\infty$, $b \neq \pm\infty$), then there exists x_{max}, x_{min} in $[a,b]$ that

$$\begin{cases} f(x) \leq f(x_{max}) & \text{for each } x \in [a,b], \\ f(x) \geq f(x_{min}) & \text{for each } x \in [a,b]. \end{cases}$$

- **Remark**. Continuous function on $[a,b]$ attains a max and a min.
- **Warning** Neither IVT nor EVT work if on (a,b) or $[a,\infty)$.

Ex Explain why there is a highest and lowest temperature everyday.

Sol.^{n} EVT on $t \in [0,24]$ (hrs), and temperature function is continuous in t.



- <u>Working Rules for Limits.</u>

Suppose $\lim_{x\to a} f(x) = L_f$, $\lim_{x\to a} g(x) = L_g$. Then:

Addition: ① $\lim_{x\to a} (f(x) \pm g(x)) = L_f \pm L_g$.

Multiplication: ② $\lim_{x\to a} (f(x)g(x)) = L_f \cdot L_g$

Division: ③ If $L_g \neq 0$, then $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L_f}{L_g}$.

Composition: ④ Let $h(x)$ be continuous at $x = L_g$. Then

$$\lim_{x\to a} h(g(x)) = h(\lim_{x\to a} g(x)) = h(L_2).$$

⑤ If $f(x) \le g(x)$ for all x close to a with $x \neq a$, then

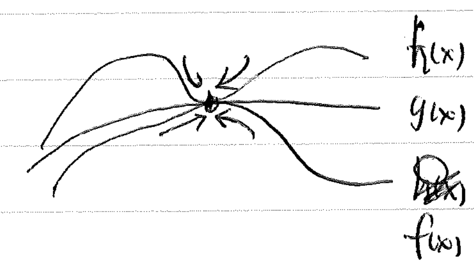
$$L_f \le L_g.$$

<u>Thm</u> (Sandwich theorem) If $f(x) \le g(x) \le h(x)$ for all x close to a with $x \neq a$, and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$



<u>Ex</u> Find the limit $\lim_{x\to\sqrt{3}} \frac{x^2+3x}{x+3}$.

<u>Sol</u>

$\lim_{x\to\sqrt{3}} x^2+3x = (\sqrt{3})^2 + 3\sqrt{3} = 3 + 3\sqrt{3}$. ($x^2+3x$ is continuous).

$\lim_{x\to\sqrt{3}} x+3 = 3+\sqrt{3}$.

So $\lim_{x\to\sqrt{3}} \frac{x^2+3x}{x+3} = \frac{3+3\sqrt{3}}{3+\sqrt{3}}$. $\square$

Ex Find $\lim_{x\to\infty} \frac{2x^2+x+3}{x^2}$.

Sol'n Note
$$\lim_{x\to\infty} \frac{2x^2+x+3}{x^2} = \lim_{x\to\infty} 2+\frac{1}{x}+\frac{3}{x^2}.$$

- $\lim_{x\to\infty} 2 = 2$
- $\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\text{large}} = \text{small} = 0.$
- $\lim_{x\to\infty} \frac{3}{x^2} = \frac{3}{\text{large}} = \text{small} = 0.$

So $\lim_{x\to\infty} 2+\frac{1}{x}+\frac{3}{x^2} = \lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{3}{x^2}$
$= 2 + 0 + 0 = 2.$ $\boxed{\Sigma}$

Thm. If f is differentiable at $x=a$, the f is continuous at $x=a$.

- Working rules for derivatives. k is a number

Scaling ① $\frac{d}{dx}(k f(x)) = k\left(\frac{d}{dx} f(x)\right)$

Addition ② $\frac{d}{dx}(f(x)+g(x)) = \left(\frac{d}{dx} f\right)(x) + \left(\frac{d}{dx} g\right)(x).$